Signal Analysis & Communication ECE 355

Ch. 2.2: CT LTI Systems

Lecture 8

25-09-2023

Ch. 2.2: CT LTI SYSTEMS

- Characterization of CT LTI systems in terms of its (unit) impulse response.

Recall - In DT LTI, the key to developing convolution sum was the sifting property, i.e.,

$$\underbrace{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}$$

any DT sig. can be expressed as superposition of scaled & shifted impulse functions

$$\delta[n] \rightarrow \boxed{S} \rightarrow h[n]$$

- Linearity & time invariancy implies that the output to $x[n]$ is:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

- Similarly, in CT LTI systems:
  - The unit impulse response $h(t)$ of a CT LTI systems is its output when unit impulse, $\delta(t)$ is the input.

$$\delta(t) \rightarrow \boxed{S} \rightarrow h(t)$$

  - We can express any sig. using sifting property of $\delta(t)$

$$x(t) = \underbrace{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau}$$

any CT sig. can be expressed as integral of scaled & shifted impulse functions

  - Linearity & time invariancy implies that the output to $x(t)$ is:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

THEOREM: For a CT LTI sys. with impulse response $h(t)$, the output, $y(t)$, to the given input $x(t)$, is:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \triangleq x(t) * h(t) \quad -(1)$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

Examples:

① $x_{(t)} = e^{-at} u_{(t)} \quad (a>0)$

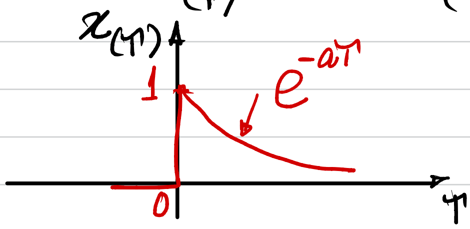
$h_{(t)} = u_{(t)}$

$y_{(t)} = ?$

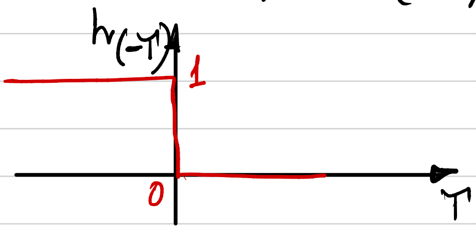
$$y_{(t)} = \int_{-\infty}^{\infty} e^{-at} u_{(\tau)} u_{(t-\tau)} d\tau$$

from eqn ① above

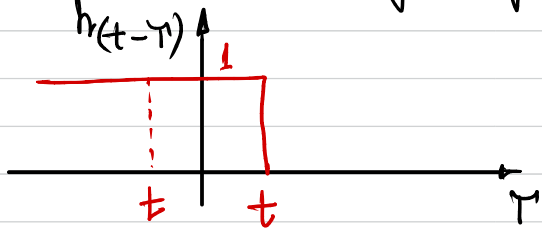
Step I Write/draw $x_{(\tau)} = e^{-a\tau} u_{(\tau)}$



Step II Write/draw $h_{(-\tau)} = u_{(-\tau)}$ ($u_{(\tau)}$ flipped)



Step III Slide $h_{(-\tau)}$ to the right by 't'



- And check for all the possible different regions of overlap/No overlap of $x_{(\tau)}$ & $h_{(t-\tau)}$ with corresponding different values of 't'.
- For example, in this considered example, we can see the two regions

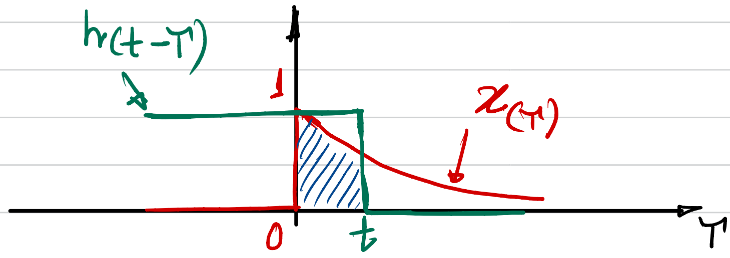
$t>0$ Overlap

$t<0$ No overlap.

Step IV a) - As per the regions of overlap in Step III, multiply $x_{(\tau)}$ & $h_{(t-\tau)}$ & integrate (eqn ①).

b) - For the non-overlapping regions, the product in eqn ① is zero.

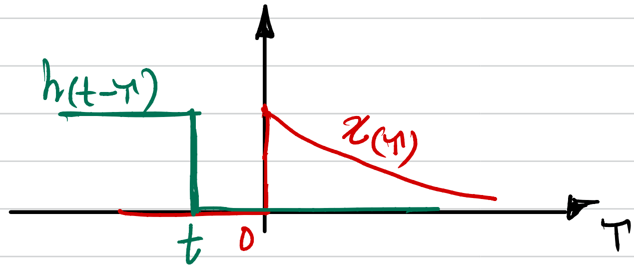
a) $\underline{t>0}$



$$y(t) = \int_0^t e^{-a\tau}(1)\, d\tau$$

$$= \left.\frac{e^{-a\tau}}{-a}\right|_0^t = \frac{1}{a}\left(1-e^{-at}\right) \quad — ②$$

b) $\underline{t<0}$



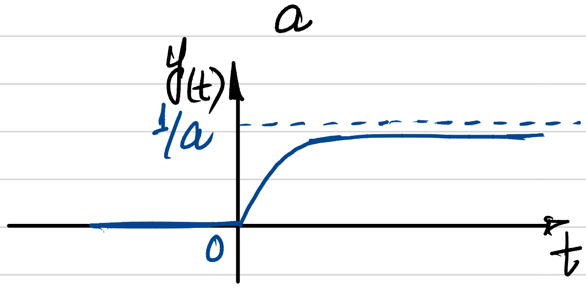
$$y(t) = 0 \qquad — ③$$

$\underline{\text{Step V}}$ Eqn ② & ③ $\Rightarrow$

$$y(t) = \begin{cases} 0 & t<0 \\ \frac{1}{a}\left(1-e^{-at}\right) & t>0 \end{cases}$$

$$= \frac{1}{a}\left(1-e^{-at}\right)u(t)$$

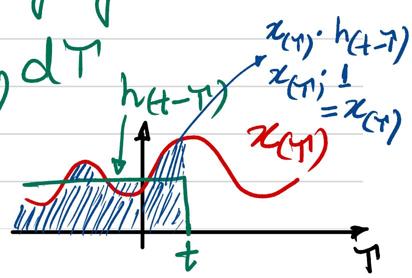
unit step to express right sided sig.



$\underline{\text{NOTE}}$: If $h(t) = u(t)$, then for any input ranging from $-\infty$ to ∞:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, d\tau$$

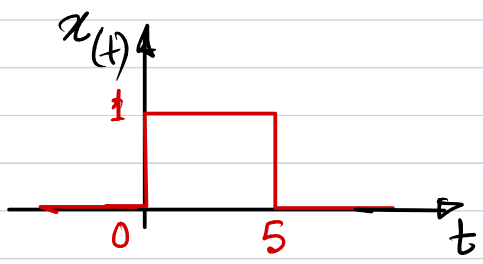


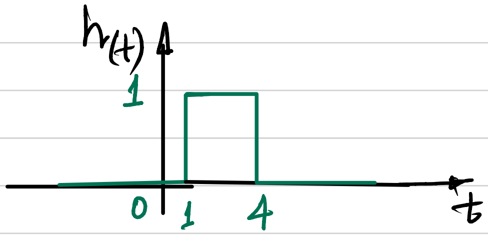
CUMULATIVE INTEGRATION

For example: if $x(t) = \delta(t)$ & $h(t) = u(t)$

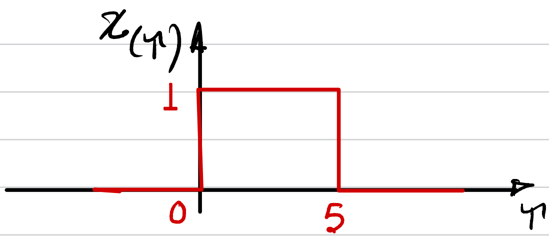
$$y(t) = \underbrace{\int_{-\infty}^{t} \delta(\tau)\, d\tau}_{= u(t)}$$

(2)

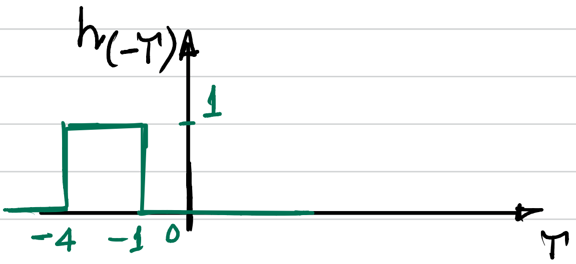




Step I

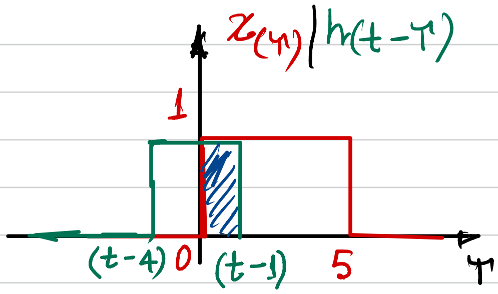


Step II



Step III

a) (i) $1 < t < 4$



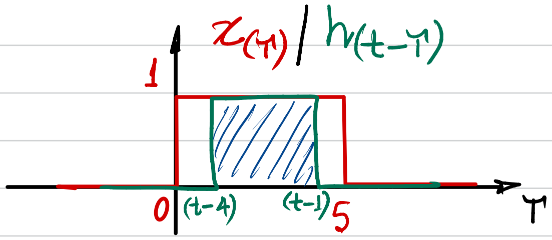
Step IV

Overlap $[0 \rightarrow (t-1)]$

$$y(t) = \int_{0}^{t-1} (1)(1)\, d\tau$$

$$= t - 1 \quad \text{— (4)}$$

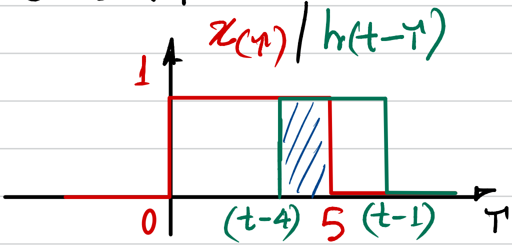
(ii) $4 < t < 6$



Overlap $[(t-4) \rightarrow (t-1)]$

$$y(t) = \int_{(t-4)}^{(t-1)} (1)(1)\,d\tau$$
$$= 3 \qquad - (5)$$

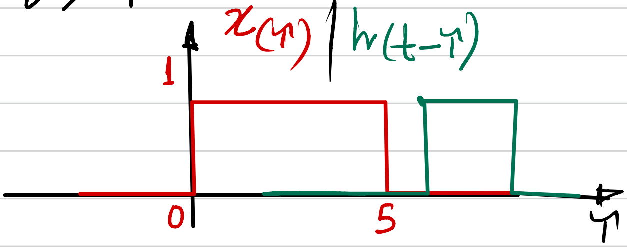
(iii) $6 < t < 9$



Overlap $[(t-4) \rightarrow 5]$

$$y(t) = \int_{t-4}^{5} (1)(1)\,d\tau$$
$$= -t+9 \qquad - (6)$$

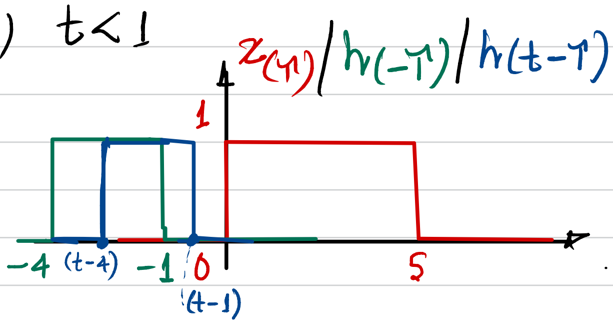
b) (i) $t > 9$



No Overlap

$$y(t) = 0 \qquad - (7)$$

Also, (ii) $t < 1$



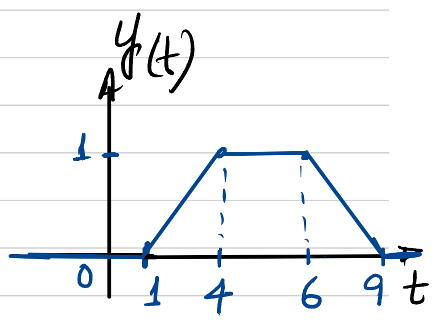
No Overlap.

$$y(t) = 0 \qquad - (8)$$

__Step V__

Combining Eqn. (4) to (8)

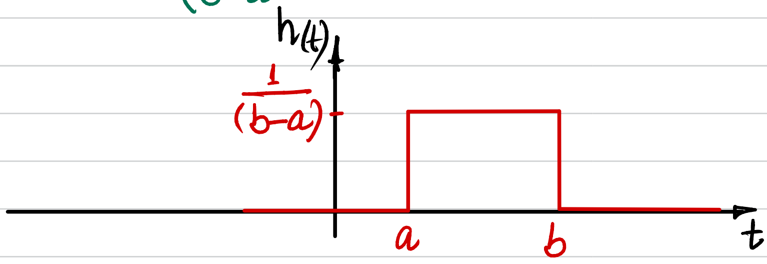
$$y(t) = \begin{cases} 0 & t < 1 \\ t-1 & 1 < t < 4 \\ 3 & 4 < t < 6 \\ -t+9 & 6 < t < 9 \\ 0 & t > 9 \end{cases}$$



REMEMBER: While performing convolution; you should look for all the regions of overlap by sliding $h(-\tau)$ to the right by 't'.

NOTE : If $h_{(t)}$ is a rectangular pulse given as:

$$h_{(t)} = \frac{1}{(b-a)}\left[u_{(t-a)} - u_{(t-b)}\right] \qquad (a<b)$$



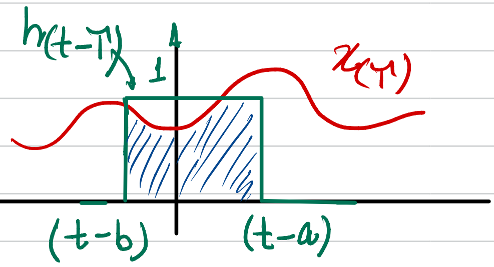
Then for any $x_{(t)}$

$$y_{(t)} = x_{(t)} * h_{(t)} = \frac{1}{(b-a)} \int_{-\infty}^{\infty} x_{(\tau)} \underbrace{\left[u_{(t-a-\tau)} - u_{(t-b-\tau)}\right]}_{} d\tau$$

For the pulse to exist, i.e., $=1$

$$\begin{cases} (t-a-\tau) > 0 \Rightarrow \tau < (t-a) \\ (t-b-\tau) < 0 \Rightarrow \tau > (t-b) \end{cases}$$

$$\Rightarrow \quad y_{(t)} = \frac{1}{(b-a)} \int_{t-b}^{t-a} x_{(\tau)} \, d\tau \qquad \text{AVERAGING.}$$



IF IMPULSE RESPONSE, $h_{(t)}$ IS A PULSE, THE OUTPUT IS THE MOVING AVERAGE OF THE INPUT.